Axisymmetric Edge-Based Finite Element Formulation For Bodies Of Revolution : Application To Dielectric Resonators

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Abstract

This paper stresses on the treatment of bodies of revolution by the finite element method (FEM) with edge elements. It clearly states an inherent difficulty on the axis of rotation specially when considering the first azimuthal mode. We propose a formulation which is not a straightforward application of standard edge elements in FEM. It takes explicitly into account a not very well known axis condition with the help of an axisymmetrical-designed edge element. Results on dielectric resonators are given and compared to measurement and to calculations from a 3D edge-based FEM code.

Introduction

Many microwave devices exhibit a rotational symmetry like cylindrical dielectric resonators or antenna horns. From a computational point of view, whenever a symmetry occurs, it is worth to benefit from it by reducing the computational domain. In axisymmetric structures, the 3D domain reduces to the 2D meridian plane. We choose here to apply a Finite element method (FEM) with edge elements enabling a robust analysis of complex inhomogeneous structures.
In the past, the FEM for axisymmetrical bodies was based mainly on the coupled azimuthal potentials [1] or the double curl equation [2] with the use of nodal elements. In these previous works, the treatment of an artificial singularity on the axis due to the curl expression in a cylindrical coordinate system has never been clearly pointed out. We will recall here the conditions that prevail on the axis and which solve the apparent singularity. The use of edge elements is now well admitted as a way to get robust algorithms. However, their application is not so straightforward particularly for the first Fourier mode case and we use instead an axisymmetrical-designed edge element [3]. Although we will derive the formulations for all the Fourier modes, we will focus on the first mode which corresponds for example to the TE_{11} mode of cylindrical waveguide. Hollow cylindrical cavities and dielectric resonators are provided to demonstrate the validity and the efficiency of the method.

Electromagnetic Fields In Axisymmetric Structures

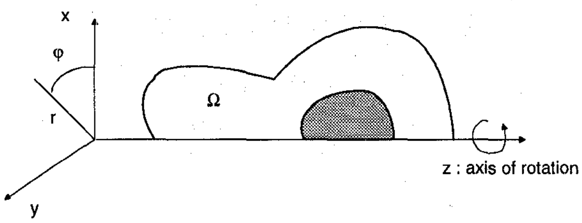

Figure 1. : Body of revolution and 2D meridional section

Let's consider a body of revolution with a cylindrical coordinate system (r, φ, z) (Fig.1). Due to the symmetry of revolution and assuming an axisymmetrical inhomogeneous medium, the electric field can be expanded in Fourier series :

$$\mathbf{E}(r,\varphi,z) = \mathbf{e}_m^0(r,z) + e_\varphi^0(r,z)\hat{\varphi} + \sum_{n\geq 1}(\mathbf{e}_m^n(r,z)\sin n\varphi + e_\varphi^n(r,z)\cos n\varphi\hat{\varphi}) + \sum_{n\geq 1}(\mathbf{e}_m^{-n}(r,z)\cos n\varphi + e_\varphi^{-n}(r,z)\sin n\varphi\hat{\varphi}) \quad (1)$$

where $\mathbf{e}_m^{\pm n}$ and $e_\varphi^{\pm n}$ are the electric field in the meridian plane and the azimuthal component of the n-th Fourier mode respectively. This expansion used in the Maxwell

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equations shows that each Fourier mode is decoupled from each other. Without loss of generality, we can only consider the modes of the form $(\mathbf{e}_m^n \sin n\varphi + e_\varphi^n \cos n\varphi\hat{\boldsymbol{\varphi}})$. For the φ-independent mode, i.e. n=0, the fields decouple further into TE $(\mathbf{e}_m^0 = 0)$ and TM $(e_\varphi^0 = 0)$ mode.

On the axis, we can distinguish [4] three kinds of conditions depending on the mode order :

1) n=0 : $\mathbf{e}_m^0$ is polarized along the axis z and $e_\varphi^0(r=0,z)=0$

2) n=1 : $\mathbf{e}_m^1$ is purely radial, $\mathbf{e}_m^1(r=0,z).\hat{\mathbf{z}}=0$, and $\mathbf{e}_m^1(r=0,z).\hat{\mathbf{r}} = e_\varphi^1(r=0,z)$

3) n>1 : all the fields vanish;
$\mathbf{e}_m^n(r=0,z)=0$, $e_\varphi^n(r=0,z)=0$.

The seemingly odd condition for n=1 expresses that the field may be polarized on the axis like the TE_{11} mode of a cylindrical waveguide. It is that situation that interests us for the theory and the application. On the axis, we have :

$$\begin{aligned}\mathbf{e}_m^1 \sin\varphi + e_\varphi^1 \cos\varphi\hat{\boldsymbol{\varphi}} &= \mathbf{e}_m^1.\hat{\mathbf{r}}\sin\varphi\hat{\mathbf{r}} + e_\varphi^1\cos\varphi\hat{\boldsymbol{\varphi}} \\ &= (\mathbf{e}_m^1.\hat{\mathbf{r}})(\sin\varphi\hat{\mathbf{r}} + \cos\varphi\hat{\boldsymbol{\varphi}}) \\ &= (\mathbf{e}_m^1.\hat{\mathbf{r}})\hat{\mathbf{y}}\end{aligned} \quad (2)$$

We will see in the sequel that these conditions are necessary to compensate the 1/r singularity which will occur when the curl of the field is taken.

Axisymmetric Finite Element Formulation

For the sake of simplicity, we will consider a cavity closed by perfect conductors, $V = \Omega\times[0,2\pi]$. Each Fourier mode verifies the weak variational form [5,6] :

$$\int_\Omega \int_0^{2\pi} \left(\frac{1}{\mu_r}\operatorname{curl}\mathbf{E}' \cdot \operatorname{curl}\mathbf{E} - k^2\varepsilon_r \mathbf{E}' \cdot \mathbf{E}\right) r\,dr\,dz\,d\varphi = 0 \quad (3)$$

The curl of the electric field for each mode is in cylindrical coordinate :

1) n=0:
$\operatorname{curl}(\mathbf{e}_m^0) = \operatorname{curl}_\varphi \mathbf{e}_m^0$ for the TM mode and

$\operatorname{curl}(e_\varphi^n\hat{\boldsymbol{\varphi}}) = \frac{1}{r}\operatorname{grad}(re_\varphi^n)\times\hat{\boldsymbol{\varphi}}$ for the TE mode (4)

2) n≥1 :

$$\operatorname{curl}(\mathbf{e}_m^n \sin n\varphi + e_\varphi^n \cos n\varphi\hat{\boldsymbol{\varphi}}) = \left(-\tfrac{n}{r}(\mathbf{e}_m^n\times\hat{\boldsymbol{\varphi}}) + (\tfrac{1}{r}e_\varphi^n\hat{\mathbf{r}} + \operatorname{grad}(e_\varphi^n))\times\hat{\boldsymbol{\varphi}}\right)\cos n\varphi + \operatorname{curl}_\varphi \mathbf{e}_m^n \sin n\varphi\hat{\boldsymbol{\varphi}} \quad (5)$$

where $\operatorname{curl}_\varphi$ and grad are the usual curl and gradient operators in a plane with Cartesian coordinates (r,z).

Equation (3) then reduces to :

1) n=0:

$$2\pi\int_\Omega \left(\frac{1}{\mu_r r^2}\operatorname{grad}(re_\varphi^0)\operatorname{grad}(re'_\varphi) - k^2\varepsilon_r e_\varphi^0 e'_\varphi\right) r\,dr\,dz = 0 \quad (6)$$

for the TE case

$$2\pi\int_\Omega \left(\frac{1}{\mu_r}\operatorname{curl}_\varphi \mathbf{e}_m^0 \operatorname{curl}_\varphi \mathbf{e}'_m - k^2\varepsilon_r \mathbf{e}_m^0\mathbf{e}'_m\right) r\,dr\,dz = 0 \quad (7)$$

for the TM case

2) n≥1:

$$\pi\int_\Omega \left(\frac{1}{\mu_r}\operatorname{curl}\mathbf{E}^n \cdot \operatorname{curl}\mathbf{E}' - k^2\varepsilon_r \mathbf{E}^n\mathbf{E}'\right) r\,dr\,dz = 0 \quad (8)$$

Let us now discuss our formulation for the n=1 case. The natural idea is to expand the electric field with edge elements, $\mathbf{w}_e$, for the meridian field $\mathbf{e}_m^1$ and nodal elements, λ_s, for the azimuthal field e_φ^1. But according to (5) and (8), this leads to undefined integrals due to a remaining 1/r singularity. Furthermore, we have to enforce the field tangential to the axis to be zero, which is very surprising because we need not do it in a 3D formulation. We could perform a numerical integration and let the variational form determines the fields. But we prefer to use a new axisymmetrical edge element [3] which takes into account the axis condition for n=1. We can see indeed that the axis condition for n=1 is necessary to overcome the singularity. The degrees of freedom (dof's) of this finite element upon a triangle are :

- Three dof's defined on the vertices of the triangle, s_i, i=1,3, as the values of the azimuthal components : $e_\varphi^1(s_i)$
- Three dof's defined on the edges, a_i, i=1,3, as the following circulations : $\int_{a_i} \frac{1}{r}(e_\varphi^1\hat{\mathbf{r}} - \mathbf{e}_m^1).\mathbf{da}$

The simplest way to describe this finite element may be to consider a change of variable, we define :

$$\mathbf{e}_m^1 = e_\varphi^n \hat{\mathbf{r}} - r\mathbf{E}_{m\varphi}. \quad (9)$$

The unknown fields will be expanded as :

$$\mathbf{E}_{m\varphi} = \sum_{e\in\{edges\}} E_e \mathbf{w}_e \quad (10)$$

$$e_\varphi = \sum_{s\in\{vertices\}} e_\varphi^s \lambda_s \quad (11)$$

where $\mathbf{w}_e$, and λ_s are edge and nodal elements respectively. The electric field of the first mode can be written as:

$$\mathbf{E}_m^1 = (e_\varphi^1 \hat{\mathbf{r}} - r\mathbf{E}_{m\varphi})\sin\varphi + e_\varphi^1 \hat{\varphi}\cos\varphi$$
$$= \Big(\sum_{s\in\{vertices\}} e_\varphi^s \lambda_s \hat{\mathbf{r}} - r \sum_{e\in\{edges\}} E_e \mathbf{w}_e\Big)\sin\varphi + \sum_{s\in\{vertices\}} e_\varphi^s \lambda_s \hat{\varphi}\cos\varphi \quad (12)$$

With (12), the apparent singularities disappear in the weak form. It is not necessary to enforce a boundary condition on the axis.

Let's now consider the modes $n \neq 1$. For the TM mode, edge elements can be directly used without any modification in equation (7). For the TE mode, it is well known that the change of variable, $e_\varphi = r\tilde{e}_\varphi$, cancels the singularity in equation (6). Unlike the authors of [3] who have generalized their n=1 finite element for n>1, which corresponds to the following change of variable, $E_{m\varphi}^n = -\frac{n}{r}\mathbf{e}_m^n + \frac{1}{r}e_\varphi^n \hat{\mathbf{r}}$, we propose, simply according to the axis condition for n>1, the use of edge and nodal elements for the change of variable : $\mathbf{e}_m^n = r\tilde{\mathbf{e}}_m^n$ and $e_\varphi^n = r\tilde{e}_\varphi^n$.

Results

Hollow cylindrical waveguide

The formulation is tested on a hollow cylindrical cavity as the analytical solutions are available. The cavity's radius and height are 5 mm and 5 mm respectively. We can see the good convergence of the results in the table I.

Dielectric resonators

The dielectric resonator is placed in a cylindrical cavity with the help of a rexolite support (Fig.2) : Dc=30 mm, Hc=21 mm, Dr=14.5 mm, Hr=9 mm, h=2 mm.

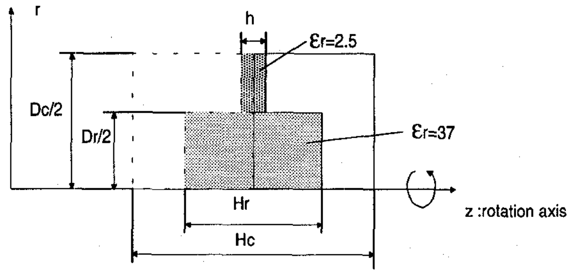


Figure 2 : Meridian cross-section of the dielectric resonator

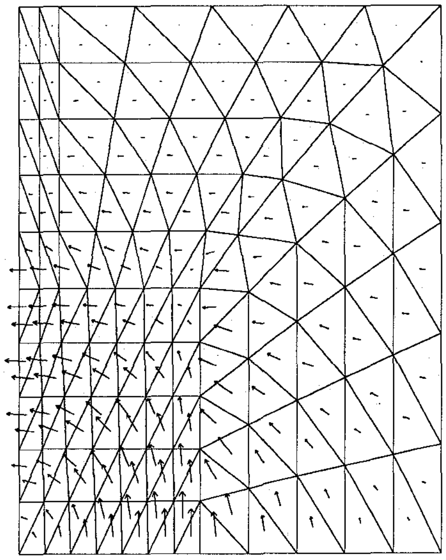
The results are reported in the table II. We can observe a very good agreement with measurement data and with numerical results calculated using a 3D FEM code with edge elements [6]. With lower than 1000 unknowns, the axisymmetric FEM provides nearly the same results as the 3D FEM one. The field patterns in a meridian plane for the two modes we called EH_I and EH_{II} are displayed in Fig.3.

Analytical results		Numerical results			
MODE	k_0 (m^{-1})	380 unknowns		880 unknowns	
		k_0 (m^{-1})	error (%)	k_0 (m^{-1})	error (%)
TE_{111}	728.45	732.10	0.50	729.70	0.17
TM_{110}	766.34	762.70	0.47	765.20	0.15
TM_{111}	991.12	992.40	0.13	991.60	0.05
TE_{121}	1237.74	1246.00	0.67	1240.00	0.18

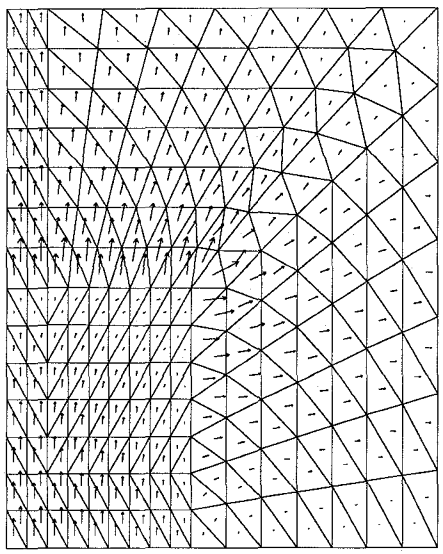
Table I : Comparison of resonance frequency for the cylindrical waveguide

MEASUREMENTS		NUMERICAL RESULTS			
		3D FEM (~5000 unknowns)		Axisymmetric (n unknowns)	
MODES	k_0 (m^{-1})	k_0 (m^{-1})	error(%)	k_0 (m^{-1})	error (%)
EHI	82.10	82.31	0.3	82.31 n=370	0.3
EHII	92.99	93.62	0.7	93.99 n=390	1.07
				93.69 n=714	0.75

Table II : Comparison of resonance frequency for the dielectric resonator



a)



b)

Figure 3 : The electric field in the meridian plane (half of the cavity) for the EHI (a) and EHII (b) modes

Conclusion

We have derived the appropriate formulation with edge elements for bodies of revolution. The first Fourier mode has been specially treated with the use of an axisymmetrical-designed edge element. The efficiency of the method has been shown with an example of a dielectric resonator.

Acknowledgments

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